ALGEBRAIC SYMBOLISM AS A CONCEPTUAL BARRIER IN LEARNING MATHEMATICS

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The use of symbolism in mathematics is probably the mostly quoted reason people use for explaining their lack of understanding and difficulties in learning mathematics. We will consider symbolism as a conceptual barrier drawing on some recent findings in historical epistemology and cognitive psychology. Instead of relying on the narrow psychological interpretation of epistemic obstacles we use the barrier for situating symbolism in the 'ontogeny recapitulates phylogeny'-debate. Drawing on a recent study within historical epistemology we show how early symbolism functioned in a way similar to concrete operational schemes. Furthermore we will discuss several studies from cognitive psychology which come to the conclusion that symbolism is not as abstract and arbitrary as one considers but often relies on perceptually organized grouping and concrete spatial relations. We will use operations on fractions to show that the reliance on concrete spatial operations also provides opportunities for teaching. We will conclude arguing that a better conceptual understanding of symbolism by teachers will prepare them for possible difficulties that students will be confronted with in the classroom.

Keywords: *Symbolism, Conceptual Barriers, Fractions, Negative Numbers*

INTRODUCTION

People who express their aversion against mathematics often quote its reliance on symbolism as the main reason for their distaste. They may be good at arithmetical calculations or geometrical constructions but once mathematical symbolism becomes necessary it puts them off. The amount of (popular) literature attempting to expound principles of mathematics without the use of symbolism is just overwhelming. For mathematics education it is an important question when and how to introduce symbolism within the curriculum. If it is introduced too early, students may lack the maturity to understand and reason symbolically. If it is introduced too late, some mathematical methods and concepts cannot be taught as they rely on symbolism.

In this paper we will consider symbolism as an obstacle in learning mathematics in the same way it has been a barrier in the historical development of mathematics. In a first section, the difference between epistemic obstacles and conceptual barriers will be discussed and related to the 'ontogeny recapitulates phylogeny'-debate. In a second section, we will provide some evidence from the history of mathematics on how the emergence of algebraic symbolism during the sixteenth century depended on earlier developments in the justification of arithmetical procedures and operations by means of graphical schemes. In a third section, evidence from cognitive psychology will be presented to argue that symbolism is not as abstract and arbitrary as one considers but often relies on perceptually organized grouping and concrete spatial relations. From this we will conclude that the unexpected concrete perceptual features of symbolism provides threats as well as opportunities for mathematics education.

EPISTEMIC OBSTACLES AND CONCEPTUAL BARRIERS IN LEARNING

When discussing students' problems with mathematical symbolism we cannot pass by the 'ontogeny recapitulates phylogeny'-debate. Some authors have taken the position that there is a strong parallel between the historical development of mathematical concepts and the acquisition of mathematical notions and concepts in the cognitive development of children. The parallel can go in both directions. In *Psychogenèse et histoire des sciences*, Piaget and Garcia (1983) identify major transitional mechanisms in geometry and algebra and relate them to the three development stages in Piaget's theory of psychogenesis. Their account of the 3000 year history of algebra identifies major stages of its historic development with the three corresponding stages of cognitive development. The first (long) period was only concerned with "solutions to specific equations. The methods used were purely empirical, trial-and-error. Each equation was treated as a separate object. This is undoubtedly an intraoperational period" (ibid, 166). The second, interoperational period started in the eighteenth century and covers Lagrange and Gauss. This period is characterized by "transformations of equations that allow the reduction of an unsolvable form to one that is solvable" (ibid.). Finally, the transoperational period originates with the group theory of Galois (1811-1832), going from equations to more abstract structures. This approach by Piaget has received little support from historians or scholars working on the crossover between history and pedagogy of mathematics.

Far more accepted is the idea that conceptual difficulties in the historical development of mathematics may be reflected in mathematics education. The most common framework for approaching this parallel is that of epistemological obstacles, a term coined in 1938 by Bachelard within the context of history of science. The term originally refers to misleading elements blocking the rational process of advancement of science. The idea was adapted by Brousseau (1976) for use in mathematics education. Brousseau attributes a positive function to epistemological obstacles within his didactical project. He considers such obstacles more as a piece of mathematical knowledge rather than a lack of knowledge. He identifies them in the history of mathematics as well as students' spontaneous models. In classroom situations they do not appear as erratic or unexpected errors, but as predictable ones. In a reaction to Brousseau, Glaeser (1981) listed a number of epistemological obstacles including the "inability to manipulate isolated negative quantities" and "the difficulty of giving meaning to isolated negative quantities". While the idea of epistemological obstacles is still at the forefront in the 'ontogeny recapitulates phylogeny'-debate, several authors have called for caution and warned that the parallelism should not be taken all too literally. Herscovics (1989) pointed out that the conditions under which concepts are thought in today's classroom are quite different from the historical conditions in which these concepts matured. Thomaidis and Tzanakis (2007) in a study on the use of the number line conclude that a strict parallelism is untenable and propose a more subtle one.

While we believe in a strong parallelism between history and education for some intrinsic difficult concepts such as symbolic reasoning, it is not necessary to resort to epistemological obstacles as a theoretical framework. We see two main objections against the concept. Within the context of history, epistemological obstacles are often viewed as blocking factors within a teleological evolution of mathematics. Such a view is considered dubious from a current perspective of mathematical practice which attributes a high degree of contingency to the development of mathematics. On the level of mathematics education, epistemological obstacles are mostly situated on the psychological level. Current scholarship takes into account a much broader contextual perspective including sociological and cultural conventions as well as norms and values. A useful alternative to epistemological obstacles is available from the history of science. An interesting phenomenon in scientific discovery is simultaneous discovery. A good example of this is the discovery of the sine law of refraction. As is now established, the sine law was discovered independently by Thomas Harriot around 1602, by Willebrord Snellius in 1621 and by René Descartes between 1626 and 1628. Descartes was the first to publish the law in his *Dioptrique* of 1637. With accurate data available since Ptolemy (2nd century AD), why did it take fifteen centuries to come to the sine law? Furthermore, why did several individuals came to the discovery within a matter of a few decades? The barrier theory by Margolis (1993) provides a framework for the understanding of such historical cases. Habits of minds govern our cognitive processes and are similar to what Polanyi called 'tacit knowledge'. They can be considered critical intuitions within a community and are therefore constitutive of a paradigm. A barrier is an entrenched habit of mind that can block a cognitive breakthrough. In relation to our subject, the cognitive barrier is symbolic reasoning. It took about two centuries from a first acceptance of isolated negative quantities to a fully symbolic treatment of the operations involved. Particularly for symbolic reasoning, the conceptual barrier functions in two directions. The conceptual distance between the prevailing arithmetical interpretation of abbaco algebra and the conflicting new ideas about symbolic reasoning in the sixteenth century determined the long historical process of difficulties. In the reverse direction, we are now so accustomed to symbolic reasoning that it becomes equally difficult to understand non-symbolic reasoning in algebraic treatises before Descartes.

In the discussion that follows, we will consider the symbolic treatment of negative quantities and operations on negative quantities as a conceptual barrier. We would like to argue that it is not so much the concept of negative quantities in itself that has led to problems and conflicts but rather the idea that negative quantities and their operations should be considered on a symbolic and abstract level. This will lead us to the conclusion that operations on negative quantities are best introduced in mathematics education within the context of symbolic algebra.

HISTORICAL EVIDENCE ON EARLY SYMBOLISM

It is difficult to pinpoint an exact date or era for the emergence of algebraic symbolism in the history of mathematics. It certainly was a convoluted process involving many actors in different cultures and covering a period of several centuries (Heeffer 2012). The most important developments took place in Renaissance Europe during the sixteenth century, culminating in the *Geometry* by Descartes (1637). This book was the first to introduce our current use of x, y and z together with exponents of unknowns. However, as argued by Heeffer (2013) the transition towards symbolic mathematics involved a process of epistemic justification and abstraction which was prepared by the European abbaco tradition flourishing in the fourteenth and fifteenth centuries. The major innovation during this period was the justification of the correctness of procedures, by which pre-symbolic algebra became liberated from constraints which accounted for the values of the quantities being operated upon. Algebraic procedures before the sixteenth century avoided negative terms (as in the rules for solving 'equations') and negative quantities as solutions of linear and quadratic problems. Earlier texts, such as the *Arithmetica* of Diophantus even avoided irrational solutions. In order to circumvent such anomalous solutions, one had to account for the values of the unknown while performing the calculations. With the justification of procedures during the abbaco period, the belief in the correctness became sufficient during the sixteenth century to allow for negative and even imaginary solutions (as in Cardano's *Ars Magnae* of 1545).

Lacking a symbolic language, abbaco masters devised graphical schemes to explain, teach and justify basic operations of arithmetic. One such scheme, shown in Figure 1, deals with operations on fractions.

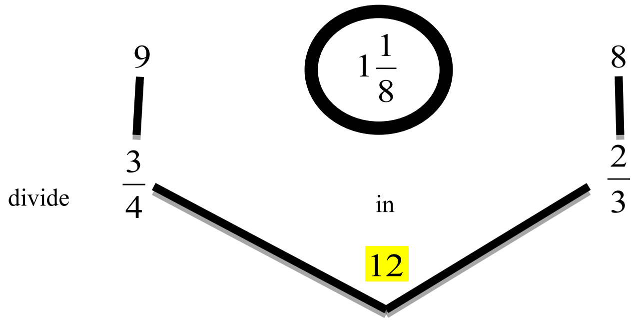

Figure 1. scheme for the division of fractions

This scheme explains how to divide one fraction by another, but the same spatial layout is also used for addition, multiplication and subtraction of fractions (Heeffer, 2013). It exploits the symbolism of the fractions bar which originated in the Maghreb region of the medieval Arab world and which was disseminated through the early the abbaco tradition (Høyrup 2010). The scheme displays intermediate results such as 9, the product 3 x 3 and 8, the product 4 x 2. The final quantity 1 1/8 is the result of dividing 9 by 8. However, the scheme also shows 12, the product of the denominators, not necessary in the calculation. This is one of the several arguments to attribute an epistemic function to this practice of adding non-discursive elements to abbaco treatises.

Schemes for the multiplication of combined fractions were extended to the multiplication of binomials by Maestro Gherardi's *Libro di ragioni* of 1328 (Arrighi 1987). Instead of bringing 12 ½ and 15 ¼ to a common denominator, he considers the two fractions as binomials (12 + ½) and (15 + ¼), each the sum of a whole number (*numero sano*) and a broken number (*rocti*).The method of crosswise multiplication, the reference to a cross (*croce*) and the non-discursive use of a configuration of schematic elements, appears frequently in abbaco texts during the following two centuries. Maestro Dardi was the first to devise a more elaborate scheme in which the four products are indicated by individual line segments (Franci, 2001). His comprehensive text on algebra, the *Aliabra argibra*, is preceded by a separate treatise dealing with operations on surds "Trattato dele regulele quale appartiene a le multiplicatione, a le divitione, a le agiuntione e a le sottratione dele radice". The multiplication $(3-\sqrt{5})(4-\sqrt{7})$ is illustrated by the scheme shown in Figure 2. It states that $(3-\sqrt{5})(4-\sqrt{7}) = 12+\sqrt{35}-\sqrt{63}-\sqrt{80}$

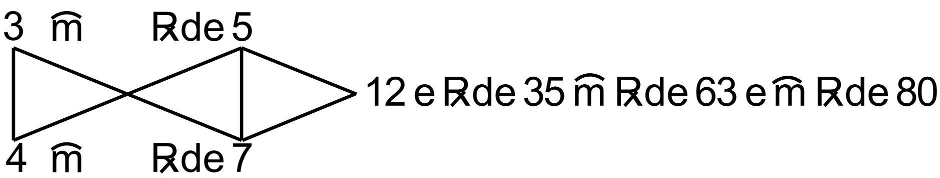

Figure 2. Dardi's scheme for multiplying surd binomials (Chigi, M.VIII.170, f. 4v)

The scheme for binomial multiplication shares all the characteristics of the schemes for operations on fractions and leads us to conclude that it serves the same function of epistemic justification of the discursive explanations.

Interestingly, in the same introduction Dardi also uses the scheme for a very different purpose, to provide a "prove" for the rules of sign (see Figure 3):

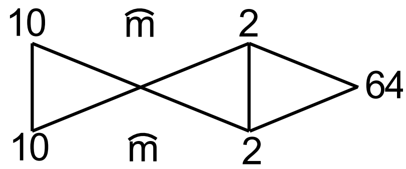

Figure 3. Dardi's use of a justification scheme for proving the rules of sign

The reasoning is as follows: we know that 8 times 8 makes 64. Therefore (10 – 2) times (10 – 2) should also result in 64. You multiply 10 by 10, this makes 100, then 10 times – 2 which is – 20 and again 10 times – 2 or – 20 leaves us with 60. The last product is (– 2)(–2) but as we have to arrive at 64, this must necessarily be +4. Therefore a negative multiplied by a negative always makes a positive.

The use of a general justification scheme for something as crucial as the laws of signs in arithmetic is quite significant. Firstly, it again shows the unifying power of such schemes. The crosswise multiplication of binomials is applied to sums and differences of natural numbers, as well as rational numbers and surd numbers. Operations on different kinds of numbers can be justified by one single scheme. Secondly, precisely because of the belief that the operations represented by such a scheme must be correct, it becomes possible to “prove” something as essential as the laws of signs. A negative multiplied by a negative must be a positive because of the validity of this scheme for multiplying binomials. Thirdly, a “proof” like this one of Dardi may seem trivial, but it is not. The reasons for the suitability of this scheme for “proving” the rules of sign go deep. If one wants to go from an arithmetic which is limited to natural numbers – as is basically the *Arithmetica* of Diophantus – to an arithmetic which includes the integers, you have to preserve the law of distribution and the law of identity for multiplication. Precisely these two laws are at play in the multiplication of binomials.

EVIDENCE FROM COGNITIVE PSYCHOLOGY

We will now look at some studies on the way symbolism functions at the cognitive level. It is a generally considered that mathematical symbolism uses an arbitrary set of signs to represent mathematical concepts and relations. Such formalist attitude to mathematics, attributed to David Hilbert, is often described by considering mathematics as “nothing but a game with meaningless symbols following meaningless rules”. Historical studies on mathematical symbolism already provide evidence against the widely held view that symbols emerged in an arbitrary way, but now additional evidence comes from research in cognitive psychology. Supported by a series of experimental studies, David Landy proposed “the revised physical symbol systems hypothesis” in which “symbols are not arbitrary, unconstrained tokens but rather are represented and processed using space and perceptually organized groups. This conception of physical symbols makes them far more constrained” than generally thought of (Landy 2007: 2039). Instead of viewing symbolic representations as lexical or sentential expressions, on a cognitive level it would be more accurate to consider them as diagrammatic and non-discursive representations. This confirms our view of early algebraic symbolism in abbaco treatises as a historical consequence of non-discursive schemes for the justification of operations (Heeffer 2013). Let us illustrate the revised physical symbol systems hypothesis by a single experiment directly relevant for mathematics education, as shown in Figure 4:

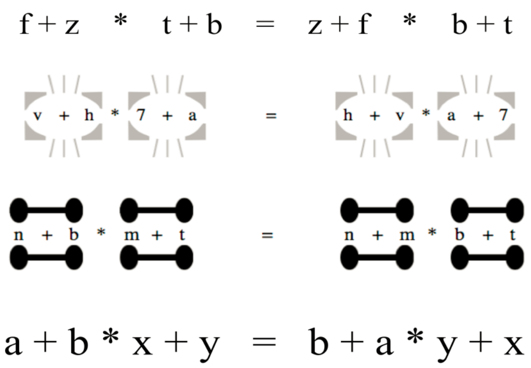

Figure 4. Sample stimuli from Landy & Goldstone (2007a) illustrating the effects of different sorts of proximity groupings in parsing symbolic expressions

Students often make mistakes against the order of operations when performing calculations or solving algebraic problems. The examples shown in Figure 4 will certainly look familiar to mathematics teachers. Mistakes in operator precedence appear to be induced not only by spatial proximities as in the top expression, but also by other types of proximity relations such as common region, connectedness, and even alphabetic proximity as shown in the lower expressions. Such experimental results on the effect of spatial organization and non-spatial grouping relations suggests that we perceive mathematical expressions in some concrete physical way rather than abstract structures that follow precise rules and syntactical relations. Other experimental setups as in Landy and Goldstone (2009) account for background motion of random dots which either impede or facilitate the solution to algebraic expressions depending on the directionality of the movement. Evidence that movement and directionality influences our interpretation of algebraic expressions provides us with strong arguments against a simple formalist view on mathematics.

CONCLUSION: THREATS AND OPPORTUNITIES FOR MATHEMATICS EDUCATION

We have considered symbolism as a conceptual barrier in the history of mathematics and as well as in learning mathematics. We have provided evidence that the earliest historical development of symbolism functioned in the same way as concrete schemes for justifying operations, such as operations on fractions and crosswise multiplication of binomials. The first explicit use of symbolism shows symbolic solutions to algebraic problems as non-discursive elements in abbaco treatises having the same epistemic function as graphical schemes for operations.

We have further referred to cognitive studies which strongly suggest that our current symbolism is not as arbitrary as often considered but instead relies on concrete spatial groupings, different layers of proximity relations, directionality and even motion. What are the consequences of these findings for the education of mathematics, specifically the introduction of symbolism?

Our current understanding of symbolism, from the history of mathematics and from cognitive psychology, provides us with a picture that poses threats as well as opportunities for the mathematics curriculum. Firstly, teachers should be aware of the fact that symbolism does not act in a completely neutral and abstract way. An insight in the way how perceptual processes direct our understanding of symbolism prepares teachers for possible mistakes and difficulties in classroom practice. The different types proximity groupings listed in Figure 4 illustrate how perceptual processes induce typical mistakes against the rules of operator precedencies. Ignoring or neglecting the perceptual basis of symbolism thus poses a threat for mathematics education.

On the other hand, the fact that symbolism is partly based on spatial organization, proximity relations, directionality and concrete physical structures also opens up new opportunities for mathematics education. Landy, Brookes and Smout (2011: 107) suggest the use of “interpretation routes” that exploit these properties of symbolism:

Rather than trying to instruct students that physical structure is irrelevant, or exclusively focusing on the intra mathematical articulation of implications, it may be possible to help students understand equations as sensible utterances by providing interpretation routes (i.e., mappings onto natural-language descriptions or imagistic models) that are both interpretable and maintain concrete relational structure. That is, rather than seeing mappings like this as a shortcut to be averted, we can see them as a route to potential understanding.

In the same way that abbaco masters employed the earliest symbols, such as the fraction bar, by devising schemes around it exploiting the spatial organization of nominators and denominators for justification as well as for teaching, so can concrete relational structures in contemporary symbolism be exploited for new and innovative ways of teaching.

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